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PION PROPERTIES FROM LATTICE QCD

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Using $12^3 \times 24$ and $16^3 \times 32$ lattices with two flavours of dynamical Wilson fermion at a variety of values of β and κ, two properties of the pion are investigated. We calculate f_π at several values of pion momentum, and estimate the second moment of the pion's quark distribution amplitude.

1. INTRODUCTION

In this paper we present the first results of a project to investigate hadronic structure based on a simulation of full QCD with two degenerate flavours of Wilson fermion[1]. So far, we have results for the pion decay constant, f_π, and for the second moment of the quark distribution amplitude in the pion[2], $\langle\xi^2\rangle$.

The 12^4 and 16^4 configurations used in this analysis were generated using the Hybrid Monte Carlo algorithm[3], and quark propagators were subsequently calculated on these lattices after replicating once in the time direction. Lattices are analysed approximately every 20 time units with an average acceptance rate of about 70%. We find that the lightest value of the pion mass attained in these simulations is approximately 600 MeV, and so we expect the effect of including dynamical quarks in these simulations to be small compared to the physical world.

It should be pointed out that the update in the Hybrid Monte Carlo runs on the 16^4 lattices initially had an error[1]. Since fixing this bug we have not noticed any systematic shift in the results. So we believe that the resultant error is of the same order as the statistical errors. The more severe limitation of this calculation is that we have very few decorrelated lattices in our statistical sample. Nevertheless, we consider it worthwhile to analyse this data since this is an exploratory calculation.

The quark propagators were computed from a "smeared" source at a fixed time slice t_0. The Wuppertal smearing method was used[4], in which the source $S(\vec{x})$ is the solution of the three dimensional gauge covariant scalar equation,

$$(-\vec{D}^2 + m_s^2)S(\vec{x}) = \delta_{\vec{x},\vec{0}}.$$

β	L	κ	N	t_π	m_π	$Z_A^{-1} f_\pi$
5.4	12	0.160	15	3	0.77(2)	0.21(1)
5.4	12	0.161	15	4	0.65(2)	0.18(2)
5.4	16	0.162	14	7	0.580(7)	0.162(8)
5.5	16	0.158	15	6	0.569(5)	0.152(7)
5.5	16	0.159	17	7	0.479(4)	0.129(6)
5.5	16	0.160	26	8	0.355(10)	0.114(5)
5.6	16	0.156	15	7	0.470(4)	0.125(6)
5.6	16	0.157	32	9	0.356(6)	0.102(4)

Table 1. Results for f_π.

The tunable parameter m_s may be regarded as a "constituent" quark mass. This gauge invariant procedure gives (for appropriately chosen m_s) a large overlap with the wave functions of low-lying hadronic states so that they dominate the behaviour of correlation functions after only a few time slices. In this calculation m_s was chosen to give a smearing radius of ≈ 4 lattice units. We plan to experiment with different smearing techniques in the future.

The parameters for the configurations used in these measurements are shown in table 1. N is the number of configurations in each sample, t_π is the time slice at which the pion dominates the π–π correlation function, and m_π is given in lattice units. The final column gives the value of f_π, discussed in the next section.

The fitting method used to extract m_π in table 1, and used throughout the calculations presented below, was as follows: we first examined the effective mass plot to determine the range of fit over which the pion saturates the 2-point correlator. We then used a single elimination "jackknife" method calculating χ^2 for each sample using the correlated

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covariance matrix. In some cases we found that the covariance matrix was almost singular and/or the fit value for the energy was very different from the effective energy. This bad behavior is due to the small statistical sample. In these cases we quote results which minimize the naive χ^2.

2. THE PION DECAY CONSTANT

The pion decay constant, f_π, is a fundamental parameter of QCD, characterizing the breaking of chiral symmetry. On the lattice with Wilson fermions, it is defined by

$$\langle 0|A_\mu(0)|\pi(\vec{p})\rangle = Z_A^{-1} f_\pi p_\mu \tag{1}$$

where A_μ is a lattice transcription of the axial current. We will use the local current

$$A_\mu(x) = \bar{\psi}(x)\gamma_5\gamma_\mu\psi(x).$$

The renormalization constant, Z_A, which relates this operator to the partially conserved continuum operator, is not known precisely. In perturbation theory, $Z_A = 0.87$ at $g^2 = 1$, but quenched simulations show this to be inaccurate[5]. We will assume a value for Z_A of around 0.8.

To determine the matrix element (1), we examine the large time behaviour of the correlators

$$K_\mu^{\rm SL}(t,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}}\langle A_\mu(\vec{x},t)J(\vec{0},0)^\dagger\rangle,$$
$$C^{\rm SS}(t,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}}\langle J(\vec{x},t)J(\vec{0},0)^\dagger\rangle, \tag{2}$$

where $J(x) \equiv \pi(x) = \bar{\psi}(x)\gamma_5\psi(x)$ or $J(x) \equiv A_4(x)$ are two local interpolating operators for the pion. The superscripts SL and SS stand for "smeared-local" and "smeared-smeared", referring to the fact that we have smeared the propagators at the source only (SL), or at both source and sink (SS).

Far from the source, these correlators are dominated by the propagation of the lightest particle having an overlap with the interpolating operators, so that

$$K_\mu^{\rm SL}(t,\vec{p}) \sim \frac{e^{-E(\vec{p})t}}{2E(\vec{p})}\langle 0|A_\mu(0)|\pi(p)\rangle\langle\pi(p)|J(0)^\dagger|0\rangle,$$
$$C^{\rm SS}(t,\vec{p}) \sim \frac{e^{-E(\vec{p})t}}{2E(\vec{p})}|\langle 0|J(0)|\pi(p)\rangle|^2,$$

where $E(\vec{p}) \approx \sqrt{\vec{p}^2 + m_\pi^2}$. For periodic boundary conditions there is a similar contribution from the propagation of the particle backwards around the lattice.

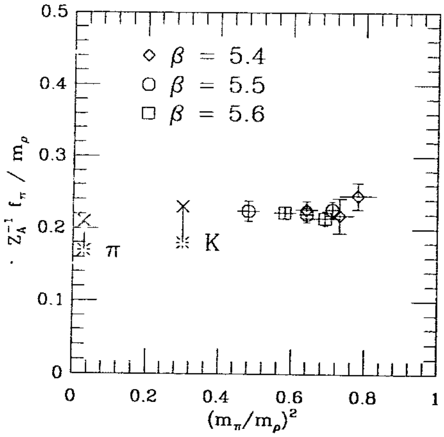

Figure 1. f_π at zero momentum.

To extract f_π from these correlators we use two methods. In method 1 we fit to the ratio of $K_\mu^{\rm SL}$ and the pion correlator $C^{\rm SS}$ to cancel the exponential fall off. The amplitude at the source which we need to remove is obtained from an independent fit to $C^{\rm SS}$. In method 2, we determine the amplitude and energy from separate single particle fits to the correlators $K_\mu^{\rm SL}$ and $C^{\rm SS}$. In this way we extract 5 parameters: $A_{\rm SL}, E_{\rm SL}, A_{\rm SS}, E_{\rm SS}$ and R where we assume that the long time behavior of the correlator is of the form $A\exp -Et$, and R is the ratio extracted directly. The final value of f_π is given by

$$\frac{1}{Z_A}f_\pi^1 = R\,\frac{E_{\rm SL}}{p_\mu}\sqrt{\frac{2A_{\rm SS}}{E_{\rm SS}}},$$
$$\frac{1}{Z_A}f_\pi^2 = \frac{2A_{\rm SL}E_{\rm SL}}{p_\mu\sqrt{2A_{\rm SS}E_{\rm SS}}},$$

for the two methods. We find that method 1 gives more stable results and the best estimate is obtained using the time component of the axial current and at zero momentum. The pseudoscalar density is marginally better as the pion operator. Our best estimate for $Z_A^{-1}f_\pi$ is given in the final column of table 1.

Figure 1 expresses our results in more physical terms. We plot $Z_A^{-1}f_\pi/m_\rho$ versus $(m_\pi/m_\rho)^2$. Also shown are the physical points for the pion and kaon (for which the relevant vector meson is $K^*(892)$), and these values rescaled by $Z_A = 0.8$. As is evident, there is good agreement between the simulation results and these rescaled points. Also $Z_A^{-1} f_\pi/m_\rho$ is essentially independent of both the quark mass and the lattice spacing.

We also measured f_π at momenta $(0,0,1)$ and $(0,1,1)$, using both space and time components of the axial current. We find these results consistent with the zero momentum values, though the errors are a little larger. This is evidence that for the small momenta we are considering, Euclidean symmetry holds to a good approximation. This can be further investigated by studying the dispersion relation $E^2 + \vec{p}^2 = m^2$. For the pion, this holds well for the the two momenta we are considering (except at the strongest coupling). For larger momenta there appear to be deviations and the errors become very large.

3. THE DISTRIBUTION AMPLITUDE

In the approach of Brodsky and Lepage[6] to exclusive hadronic processes at large momentum transfer (high Q^2), the scattering amplitude is approximated by the convolution of a perturbatively calculable hard scattering amplitude with wave functions describing the overlap of the participating hadrons with their lowest Fock state. These wave functions - the quark distribution amplitudes in the hadron - are hypothesized to contain the non-perturbative physics.

The distribution amplitude for the pion, ϕ, depends on $\xi = x_q - x_{\bar{q}}$, where x_q and $x_{\bar{q}}$ are the fractional light-cone momenta carried by the quark and antiquark respectively, as well as on the renormalization scale Q^2 at which the wave function is defined. The general features of $\phi(\xi, Q^2)$ can be deduced from its moments $\langle \xi^n \rangle = \int_{-1}^{1} d\xi\, \xi^n \phi(\xi, Q^2)$. ϕ is normalized such that $\langle \xi^0 \rangle = 1$, and odd moments vanish by parity. The moments are related to the matrix elements of the local operators

$$O^{(n)}_{\mu_0\mu_1\ldots\mu_n} = i^n \bar{\psi}\gamma_{\mu_0}\gamma_5 \overleftrightarrow{D}_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\psi,$$

through the operator product expansion, giving

$$\langle 0|O^{(n)}_{\mu_0\mu_1\ldots\mu_n}|\pi(p)\rangle = f_\pi p_{\mu_0} p_{\mu_1}\cdots p_{\mu_n}\langle \xi^n \rangle. \quad (3)$$

The operators are understood to be traceless, and in this study we symmetrize over Lorentz indices, though eq. (3) holds in any case, as a consequence of Lorentz covariance. The Q^2 dependence of these matrix elements is calculable within perturbation theory, and consequently the large-Q^2 form of the wave function is known, giving $\langle \xi^2 \rangle = 0.2$. At accessible energy scales non-perturbative techniques are necessary. We mention previous calculations using two methods: sum rules and lattice QCD.

Sum rule estimates[7] give the surprising result $\langle \xi^2 \rangle \approx 0.4$ at $Q^2 = (0.5\ \text{GeV})^2$, indicating a small amplitude for momentum to be shared evenly between constituents. Two (quenched) lattice calculations gave the results $\langle \xi^2 \rangle_{\text{latt}} = 0.26 \pm 0.13$[8], and $\langle \xi^2 \rangle_{\text{latt}} = 0.30 \pm 0.13$[9]. The subscript indicates that these are bare lattice numbers. To compare to continuum measurements we need to include a renormalization constant which, at $g^2 \approx 1, Q^2 = a^{-2} \approx (2\ \text{GeV})^2$, will probably increase these results by 1.2–1.4 (though the exact value is not known). Within errors these lattice calculations agree with the sum rule estimates, but the errors are very large.

To extract $\langle \xi^2 \rangle$ we study the ratio of two correlators:

$$K^{SL}_{\mu\nu\rho}(t,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle O^{(2)}_{\mu\nu\rho}(\vec{x},t) J(\vec{0},0)^\dagger \rangle,$$

and K^{SL}_μ of eq. (2). Because of the absence of Euclidean symmetry we must be careful to choose an operator, $O^{(2)}$, which does not mix with lower dimensional operators[8]. Satisfactory choices are $O^{(2)}_{433} - 1/2(O^{(2)}_{411} + O^{(2)}_{422})$ and $O^{(2)}_{423}$. For each of these we can divide by either K^{SL}_4 or K^{SL}_3, and we denote the four possible ratios by $R^{433}_{4,3}$ and $R^{423}_{4,3}$. Away from source, these should be constant in time:

$$R^{433}_{4,3} = p_3 p_{3,4} \langle \xi^2 \rangle_{\text{latt}},$$
$$R^{423}_{4,3} = p_2 p_{3,4} \langle \xi^2 \rangle_{\text{latt}}.$$

The minimal momenta for which these do not vanish are $(0,0,1)$ and $(0,1,1)$ respectively. From our analysis of f_π we believe the granularity of the lattice is not a severe problem for these momenta.

A greater problem is having sufficient statistics to extract a signal. We believe the signal we obtain is significantly better than in previous lattice studies (though still poor), and attribute this improvement to the use of smeared propagators.

Our results for $\langle \xi^2 \rangle$ are given in table 2. In figure 2 we plot our best results, those for $R^{433}_{4,3}$, versus $(m_\pi/m_\rho)^2$. From this graph it is difficult to draw any conclusions about the behaviour of $\langle \xi^2 \rangle$ with the quark mass. Assuming weak dependence on the quark mass, we average our data to find

$$\langle \xi^2 \rangle_{\text{latt}} = 0.10 \text{ using } R^{433}_{4,3},$$
$$\langle \xi^2 \rangle_{\text{latt}} = 0.11 \text{ using } R^{423}_{4,3}.$$

R_4^{433}/p_3p_3	R_3^{433}/p_3p_4	R_4^{423}/p_2p_3	R_3^{423}/p_2p_4
0.074(9)	0.103(6)	0.092(11)	0.119(13)
0.085(9)	0.133(6)	0.099(13)	0.147(10)
0.092(4)	0.092(11)	0.105(19)	0.122(8)
0.081(11)	0.062(18)	0.103(19)	0.056(23)
0.109(4)	0.139(19)	0.123(6)	0.148(11)
0.101(15)	0.130(18)	0.105(15)	0.127(23)
0.086(10)	0.138(9)	0.084(9)	0.131(21)
0.097(10)	0.099(16)	0.100(13)	0.102(17)

Table 2. Results for $\langle\xi^2\rangle$. Rows correspond to the same parameter values as in table 1.

We do not quote an error, leaving it to the reader to decide on the reliablility of our results by examining table 2.

These results are much lower than the previous lattice estimates, but the errors quoted in these papers were very large. More significant is the discrepancy between our result and sum rule calculations[7]. Even allowing for a large error in our result, it would require a very large renormalization constant and/or strong quark mass dependence to reconcile the two.

4. SUMMARY AND CONCLUSIONS

Using a variety of lattice parameters, and including the effects of two flavours of Wilson fermions, we have measured f_π and the second moment of the quark distribution amplitude in the pion, $\langle\xi^2\rangle$. Both of these measurements require that we extract the residue of the pion's contribution to various correlators. The use of "smeared" sources has allowed this to be done more cleanly than before.

For f_π we see approximate momentum independence for the three momenta considered. The actual value of f_π for all three values of β agrees well with the physical value assuming a lattice renormalization constant for the axial current of around 0.8.

Our measurements of $\langle\xi^2\rangle$ suggest that the fractional light-cone momentum of the pion is divided more evenly between the component quark and antiquark than previous studies, using sum-rules and quenched lattice QCD, have indicated.

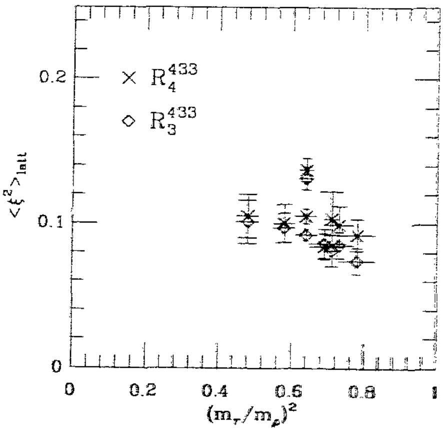

Figure 2. $\langle\xi^2\rangle$ from $R_{4,3}^{433}$.

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